ADDITIVITY FOR BEGINNERS

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Abstract. I recall first parts of Florian's talk since a month has passed. Then I explain how operads become graded Lie algebras and operads with multiplication become Gerstenhaber algebras up to homotopy. This is the first instance of the additivity theorem which is my main aim in this seminar: E_m-algebras in E_n-algebras are E_{m+n}-algebras. This means many things depending on the context, and I think $m = n = 1$ alone subsumes the Echmann–Hilton argument and our story today. Not everything in these notes is fully understood and worked out, but instead of writing out all the different theories (symmetric, nonsymmetric etcetc) separately I tried to sketch how I think a unified framework seems to look.

Contents

1. Operads with multiplication

In this first section I recall and generalise various definitions introduced by Florian last time. Afterwards I define operads with multiplication, which is new material.

1.1. **Categorical setup.**

1.1.1. $\mathscr{B}$. Throughout, $(\mathscr{B}, \otimes, 1, \mathfrak{s})$ is some symmetric monoidal base category. I will freely assume that $\mathscr{B}$ has additional properties and structures, e.g. that it is closed and has funtorial (co)limits (i.e. that it is a *cosmos*). All other categories are tacitly assumed to be $\mathscr{B}$-enriched. I usually pretend that $\mathscr{B}$ is one of the following examples:

(1) $\mathbf{Set}$ (sets, Florian just considered this case),
(2) $\mathbf{Top}$ (topological spaces) $\rightsquigarrow$ “topological operads”,
(3) $\mathbf{Mod}_{\mathbb{K}}$ (vector spaces) $\rightsquigarrow$ “algebraic operads”.

Thus I use elements to represent morphisms in $\mathscr{B}$; in particular, the symmetry $\mathfrak{s}_{V,W}\colon V \otimes W \to W \otimes V$ will be represented by writing $v \otimes w \mapsto w \otimes v$. I also pretend that all monoidal categories are strict.

1.1.2. $\mathscr{X}$. We will also use graded vector spaces and to incorporate this we fix a symmetric monoidal categroy $(\mathscr{X}, +, 0)$ whose objects play the role of the possible degrees of objects. Three key examples we will use are:

(1) $\mathbf{1} \rightsquigarrow$ “ungraded case”,
(2) $\mathbb{N} \rightsquigarrow$ “positively graded case”,
(3) $\mathbb{Z} \rightsquigarrow$ “graded case”.

Here $\mathbb{Z}$ are the integers viewed as a discrete category. So for $\mathscr{B} = \mathbf{Set}$, there are no morphisms $i \to j$ in $\mathbb{Z}$ unless $i = j$ in which case there is one, the identity. For general $\mathscr{B}$, we have

$$\mathbb{Z}(i, j) \coloneqq \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases}$$

where 0 is the initial object in $\mathscr{B}$. Good remark by Zbiggi: if $\otimes$ is cocontinuous (e.g. as $\mathscr{B}$ is closed), $0 \otimes 0 \cong 0$. Similarly, $\mathbb{N}$ are the natural numbers viewed as discrete category, while $\mathbf{1}$ denotes the terminal category, that is, the full subcategory of $\mathbb{Z}$ with the one object 0. I was tempted to add another nondiscrete example which leads to the “differentially graded case” but I felt this all gets out of hand a bit.

1.1.3. $\mathscr{C}$. There will be three other categories to discuss. We will define operads in the functor category

$$\mathscr{C} \coloneqq [\mathscr{X}, \mathscr{B}],$$

which for $\mathcal{I} = \mathbf{1}$ is $\mathcal{B}$ and for $\mathcal{I} = \mathbb{Z}$ is the category $\mathbf{Gr}(\mathcal{B})$ of *($\mathbb{Z}$-)graded objects in $\mathcal{B}$*. In particular, we obtain for $\mathcal{I} = \mathbb{Z}, \mathcal{B} = \mathbf{Mod}_{\mathbb{K}}$ the category $\mathbf{GrMod}_{\mathbb{K}}$ of graded vector spaces. In such concrete settings I will write

$$|x| := n, \quad x \in \mathsf{V}(n)$$

and say $x \in \mathsf{V}$ is *homogeeous of degre n.*

The category $\mathcal{C}$ is monoidal with respect to the *Day convolution* that I also denote by $\otimes$. In general, this is the coend

$$(\mathsf{V} \otimes \mathsf{W})(n) := \sum^{i,j \in \mathcal{I}} \mathcal{I}(i+j, n) \otimes \mathsf{V}(i) \otimes \mathsf{W}(j),$$

but when $\mathcal{I}$ is discrete this just means

$$(\mathsf{V} \otimes \mathsf{W})(n) = \bigoplus_{i+j=n} \mathsf{V}(i) \otimes \mathsf{W}(j)$$

so that the unit object is

$$(\delta_0)(n) := \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases}$$

The *desuspension* of $\mathsf{V} \in \mathcal{C}$ is $\mathsf{s}^{-1}\mathsf{V} \in \mathcal{C}$ given by

$$(\mathsf{s}^{-1}\mathsf{V})(i) := \mathsf{V}(i+1).$$

In the cases I care about there should also be an internal hom, $[\mathsf{V}, \mathsf{W}](n) = \mathcal{C}(\mathsf{s}^{-n}\mathsf{V}, \mathsf{W})$, might matter, don't know yet. I am at this point also reminded about discussions with Stephanie, maybe we want the internal and the external hom in $\mathcal{B}$ to agree in a sense one can make precise.

1.1.4. t. I haven't worked this out but will put this on the list of BSc/MSc topics: $\mathcal{C}$ inherits a symmetry from $\mathcal{B}$, but we want to generalise this to a twisted symmetry that I will ruthlessly represent by the formula

$$t_{\mathsf{V},\mathsf{W}} \colon \mathsf{V} \otimes \mathsf{W} \to \mathsf{W} \otimes \mathsf{V}, \quad x \otimes y \mapsto y \otimes x$$

while I should really write

$$t_{\mathsf{V},\mathsf{W}} \colon \mathsf{V} \otimes \mathsf{W} \to \mathsf{W} \otimes \mathsf{V}, \quad x \otimes y \mapsto (x_{(-1)} \triangleright y) \otimes x_{(0)}.$$

Here $\mathcal{I}$ acts and coacts somehow; maybe $\mathcal{C}$ doesn't even have to be $[\mathcal{I}, \mathcal{B}]$ but is just anything in the symmetric centre of the category of $\mathcal{I}$-module categories? A pure grading by a discrete $\mathcal{I}$ gives

$$t_{\mathsf{V},\mathsf{W}}\colon \mathsf{V}(i)\otimes\mathsf{W}(j)\to\mathsf{W}(j)\otimes\mathsf{V}(i),\quad x\otimes y\mapsto i\triangleright y\otimes x$$

and for $\mathcal{I}=\mathbb{Z}$ this becomes

$$t_{\mathsf{V},\mathsf{W}}\colon \mathsf{V}(i)\otimes\mathsf{W}(j)\to\mathsf{W}(j)\otimes\mathsf{V}(i),\quad x\otimes y\mapsto r^i(y)\otimes x$$

for some natural (parity, involution, ribbon...?) operator r which for $\mathcal{B}=\mathbf{Mod}_{\mathbb{K}}$ can be taken to be either the identity or to be the one that acts on $\mathsf{W}(j)$ by $(-1)^j$ so that

$$t_{\mathsf{V},\mathsf{W}}\colon \mathsf{V}(i)\otimes\mathsf{W}(j)\to\mathsf{W}(j)\otimes\mathsf{V}(i),\quad x\otimes y\mapsto(-1)^{ij}y\otimes x.$$

This makes sense whenever $\mathcal{B}$ carries an involution that I represent by $x\mapsto -x$ and that should satisfy things such as $-(x\otimes y)=(-x)\otimes y=x\otimes(-y)$.

1.1.5. $\mathcal{D}$. Our main aim is to define what it means to add an algebraic structure to an object of some monoidal category $(\mathcal{D},\star,\mathrm{I})$. The basic example is $\mathcal{D}=\mathcal{C}$, but another good one is $\mathcal{B}=\mathcal{C}=\mathbf{Mod}_{\mathbb{K}}$ and $\mathcal{D}=\mathbf{Mod}_{R^e}$, where R is a $\mathbb{K}$-algebra and $R^e=R\otimes R^{\mathrm{op}}$ is its *enveloping algebra* so that $\mathcal{D}$ is the category of R-bimodules with symmetric action of $\mathbb{K}$.

In one approach to operads we will need $\mathcal{D}$ to be a $\mathcal{C}$-module category via

$$\cdot\colon \mathcal{C}\times\mathcal{D}\to\mathcal{D}$$

and we then assume that $-\cdot X$ is left adjoint to $\mathcal{D}(X,-)$,

$$\mathcal{D}(C\cdot X,Y)\cong\mathcal{C}(C,\mathcal{D}(X,Y)),\quad C\in\mathcal{C},X,Y\in\mathcal{D}$$

as objects in $\mathcal{B}$. For example, when $\mathcal{C}=\mathbf{Mod}_{\mathbb{K}}$ and $\mathcal{D}=\mathbf{Mod}_{R^e}$, we are in business with

$$C\cdot X=C\otimes X,$$

being the tensor product over $\mathbb{K}$ (R-actions just on X).

1.1.6. $\mathcal{N}$. Finally, we fix a monoidal category $(\mathcal{N},+,0)$ whose monoid of objects are the natural numbers under addition, and we assume that $\mathcal{N}$ acts by natural symmetries on $\star$. Again, three examples are sufficient for me that I will define in detail in a moment, namely

(1) $\mathbb{N}$ (dafault) $\rightsquigarrow$ “planar (= nonsymmetric) operads”,
(2) $\mathbb{S}$ (Florian’s default) $\rightsquigarrow$ “symmetric operads”,

(3) $\mathbb{F} \rightsquigarrow$ "cartesian operads (= clones = Lawvere theories)".

AFAIK, Kelly (Max, not Maggs) taught mankind how to assign to $\mathcal{N}$ a 2-monad $\mathsf{T}_{\mathcal{N}}$ on $\mathbf{Cat}$, and the assumption is that $\mathcal{D}$ is a 2-algebra over this 2-monad. If $\mathcal{N} = \mathbb{N}$, this means that $\mathcal{D}$ is any monoidal category. If $\mathcal{N} = \mathbb{S}$, $\mathcal{D}$ is a symmetric monoidal category, and if $\mathcal{N} = \mathbb{F}$, $\mathcal{D}$ is a cartesian category. In general, $\mathcal{N}$ can be recovered as the free monoidal category $\mathsf{T}_{\mathcal{N}}(\mathbf{1})$ of the specified type on a single gnerator. Other interesting choices of $\mathcal{N}$ yield braided monoidal categories, closed symmetric monoidal categories, and semicartesian categories (monoidal categories in which I is terminal). In fact, one can even go beyond the case in which the objects of $\mathcal{N}$ are just the natural numbers, see Example 1.2.12 below, but I think for us this is good enough.

1.1.7. *$\mathbb{F}$ and $\mathbb{S}$.* By $\mathbb{F}$ I denote the skeleton of $\mathbf{FinSet}$ which has as objects tha sets $i \coloneqq [i-1] \coloneqq \{0, \dots, i-1\}$ with

$$\mathbb{F}(i, j) \coloneqq \mathbf{Set}(i, j),$$

the set of all set maps $i \to j$. By $\mathbb{S}$, I denote the *permutation category* which is the core of $\mathbb{F}$,

$$\mathbb{S}(i, j) \coloneqq \begin{cases} 0 & i \neq j, \\ \mathsf{S}_i & i = j. \end{cases}$$

Of course we need to jazz these up to objects of $\mathcal{B}$. When $\mathcal{B} = \mathbf{Mod}_{\mathbb{K}}$ this means that S_i is the group algebra $\mathbb{K}S_i$ of the symmetric group S_i. This is another loose end of this talk where I forced myself to not work things out.

1.2. **Operads.**

1.2.1. *Approach I.* An *operad (in $\mathcal{C}$ of type $\mathcal{N}$)* is a functor $\mathsf{O}\colon \mathcal{N}^{\mathrm{op}} \to \mathcal{C}$ together with morphisms

$$\circ_{p,i}\colon \mathsf{O}(p) \otimes \mathsf{O}(q) \to \mathsf{O}(p+q-1), \quad i = 1, \dots, p$$

satisyfing a unitality axiom (there is a unary operation $\mathrm{id} \in \mathsf{O}(1)$ that is an identity for all $\circ_{p,i}$), an associativity axiom

$$(\varphi \circ_{p,i} \psi) \circ_{p+q-1,j} \chi = \begin{cases} (\varphi \circ_{p,j} \chi) \circ_{p+r-1,i+r-1} \psi & j < i, \\ \varphi \circ_{p,i} (\psi \circ_{q,j-i+1} \chi) & i \le j \le i+q+1, \\ (\varphi \circ_{p,j-q+1} \chi) \circ_{p+r-1,i} \psi & i+q \le j, \end{cases}$$

and an axiom that expresses the compatibility with morphisms in $\mathcal{N}$. These axioms are readily derived if one represents $\circ_{P,i}$ pictorially as follows:

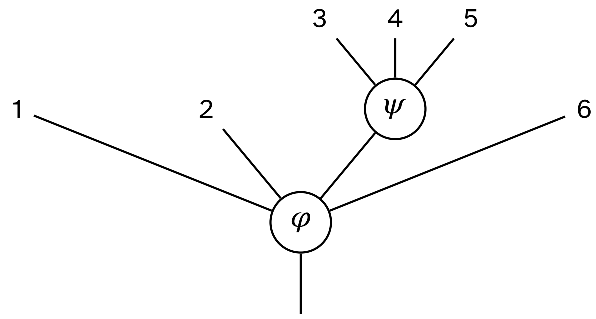

Figure 1. $\varphi \circ_{4,3} \psi$ $(p = 4, q = 3)$

We call $\varphi \in \mathrm{O}(n)$ an *n-ary operation in* O.

1.2.2. *Remark.* Note that in the associativity axiom, the symmetry t of $\mathscr{C}$ enters when ψ and χ change places: just as you need a symmetric monoidal category to enrich monoidal categories, you need it to define operads. So when $\mathscr{C} = \mathbf{GrMod}_{\mathbb{K}}$ is the category of graded vector spaces, the above formulas suppress signs $(-1)^{|\psi||\chi|}$ in cases 1 and 3. I think there are some truly Wittgensteinian topics to be thought about here, with fading eyesight and hearing I become a more and more pure algebraist, I do not process pictures or melodies (see The Arrival), but a linear discrete stream of symbols. Let's move on I'd say, but I will get back to this briefly in 1.2.8.

1.2.3. *Caveat.* Some people draw resp. read pictures upside down, some from right to left, some do both and then explicit formulas for the axioms change, e.g. the indices in the above associativity axiom. Besides this convention on how to order the inputs of an n-ary operation, there is the convention on how to order the operations that are composed (whether the picture define $\varphi \circ_{4,3} \psi$ or $\psi \circ_{4,3} \varphi$). In this, we stick to the traditional convention on compositions of functions. The abstract Approaches III - VI below avoid these troubles.

1.2.4. *Approach II.* Equivalently, one may define an operad as a functor $\mathrm{O} \colon \mathcal{N}^{\mathrm{op}} \to \mathscr{C}$ plus morphisms

$$\circ_{i_1,\dots,i_n} \colon \mathrm{O}(n) \otimes \mathrm{O}(i_1) \otimes \cdots \otimes \mathrm{O}(i_n) \to \mathrm{O}(i_1 + \cdots + i_n)$$

whose axioms are derived from pictures such as the following:

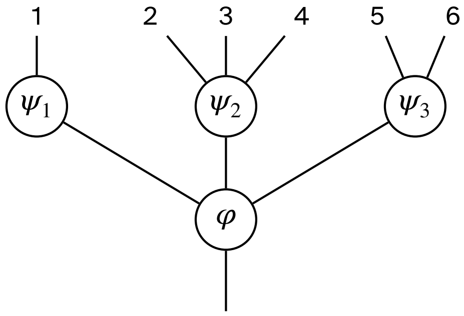

Figure 2. $\circ_{1,3,2}(\varphi \otimes \psi_1 \otimes \psi_2 \otimes \psi_3)$

1.2.5. *Example*. The *endomorphism operad* End_X of an object X in $\mathscr{D}$ has

$$\mathsf{End}_X(n) := \mathscr{D}(X^{\star n}, X)$$

with

$$\varphi \circ_{p,i} \psi := \varphi \circ (\mathrm{id}_{X^{\star i-1}} \star \psi \star \mathrm{id}_{X^{\star p-i}}).$$

If O is any operad, then an O-*algebra structure* on X is an operad morphism $\alpha\colon \mathsf{O} \to \mathsf{End}_X$, that is, a natural transformation of functors $\mathscr{N}^{\mathrm{op}} \to \mathscr{C}$ that is compatible with the $\circ_{p,i}$.

1.2.6. *Example*. For $\mathscr{N} = \mathbb{N}$, the *planar associative operad* $\mathsf{Ass}^{\mathbb{N}}$ is given by setting for all n

$$\mathsf{Ass}^{\mathbb{N}}(n) := 1,$$

with all $\circ_{p,i}$ being the canonical isomorphism $1 \otimes 1 \cong 1$. An $\mathsf{Ass}^{\mathbb{N}}$-algebra is a unital associative algebra (a monoid) in $\mathscr{D}$.

1.2.7. *Example*. When $\mathscr{N} = \mathbb{S}$ and we make all S_n act trivially on 1, we would instead call this the *commutative operad*

$$\mathsf{Comm}(n) := 1.$$

That is, if we forget the trivial symmetry, the symmetric operad Comm becomes the planar associative operad. But we don't, and that a Comm-algebra structure $\alpha\colon \mathsf{Comm} \to \mathsf{End}_X$ is in particular a natural transformation of functors $\mathbb{S}^{\mathrm{op}} \to \mathscr{C}$ shows that the Comm-algebras are precisely the commutative algebras in $\mathscr{D}$; this structure could not have been defined when $\mathscr{N} = \mathbb{N}$. However, we may now define the *symmetric associative operad* $\mathsf{Ass}^{\mathbb{S}}$ by

$$\mathsf{Ass}^{\mathbb{S}}(n) := \mathsf{S}_n$$

and then $\mathsf{Ass}^{\mathbb{S}}$-algebras are again just algebras in $\mathscr{D}$. Note Florian was considering this symmetric associative operad, not the planar one.

1.2.8. *Remark.* There are also *cyclic* and *modular* operads, but this is about duality in $\mathscr{D}$, and about the spatial arrangement of the pictures that represent compositions that could be drawn not in the plane but on some oriented compact smooth manifold of dimension 2 (compare spherical and cylindrical monoidal categories).

1.2.9. *Approach III.* Recall that we assume

$$\mathscr{D}(C \cdot X, Y) \cong \mathscr{C}(C, \mathscr{D}(X, Y)), \quad C \in \mathscr{C}, X, Y \in \mathscr{D}.$$

In a concrete setting, a sequence of morphisms

$$\alpha_n \colon C_n \to \mathscr{D}(X^{\star n}, X) = \mathsf{End}_X(n), \quad n \geq 0$$

in $\mathscr{C}$ thus corresponds to a sequence of morphisms

$$\hat{\alpha}_n \colon C_n \cdot X^{\star n} \to X$$

in $\mathscr{D}$. If $C_n = \mathsf{O}(n)$ for a functor $\mathsf{O}\colon \mathscr{N}^{\mathrm{op}} \to \mathscr{C}$ and the α_n are the components of a natural transformation, then the $\hat{\alpha}_n$ assemble into a single morphism $\hat{\alpha}\colon \hat{\mathsf{O}}(X) \to X$ in $\mathscr{D}$, where $\hat{\mathsf{O}}$ is the endofunctor

$$\hat{\mathsf{O}}\colon \mathscr{D} \to \mathscr{D}, \quad X \mapsto \sum^{n \in \mathscr{N}} \mathsf{O}(n) \cdot X^{\star n},$$

where the right hand side is a coend. Joyal (I think) called this an *analytic* functor as it looks like a power series. In the world of algebraic operads, O would be called an $\mathbb{S}$-module and $\hat{\mathsf{O}}$ the associated Schur functor [4].

When O is an operad, then $\hat{\mathsf{O}}$ is a monad, and in the down to earth cases I care about this is an if and only if. So operads are (or correspond to) analytic monads and O-algebras are the algebras over these monads.

1.2.10. *Approach IV.* One can also carry the above over to a monoidal product on the functor category $[\mathscr{N}^{\mathrm{op}}, \mathscr{C}]$ (the *composite* or *substitution product* $\circ$) and then an operad is simply a monoid therein. This has the advantage that we do not need to introduce any category $\mathscr{D}$ at all but study the operad in its own right. However, I am more interested in O-algebras hence move on.

1.2.11. *Example.* For $\mathsf{O} = \mathsf{Ass}^{\mathbb{N}}$, $\hat{\mathsf{O}}(X)$ is the tensor algebra $\bigoplus_{n\geq 0} X^{\star n}$ which is the free associative algebra on X. For $\mathsf{O} = \mathsf{Comm}$, $\hat{\mathsf{O}}(X)$ is the free commutative algebra on X, that is, the symmetric algebra. The free Lie algebra is a more subtle topic, see e.g [10] which also contains a good account on the substitution product.

1.2.12. *Example.* In [9], Shulman points out that you can go way beyond our setting: That an operation has a finite number of inputs is irrelevant and at least for $\mathscr{C} = \mathbf{Set}$ he claims you could actually also take $\mathscr{N} = \mathscr{C}^{\mathrm{op}}$. Then $[\mathscr{N}^{\mathrm{op}}, \mathscr{C}]$ is the category of endofunctors on $\mathscr{C}$ and $\circ$ becomes just the composition, so an operad is a monad on $\mathscr{C}$. This is a bit as with rings vs. algebras: rings are special $\mathbb{K}$-algebras (namely $\mathbb{K} = \mathbb{Z}$) and not the other way round.

1.2.13. *Approach V.* Florian worked from the start with *multicategories* (= coloured operads). Then an operad is just a multicategory with a single object. I won't need this, but recall that to any monoidal category $\mathscr{D}$ Florian associated the multicategory $\mathsf{End}_{\mathscr{D}}$ *represented by* $\mathscr{D}$ (the endomorphism operad is the case of a monoidal category that is monoidally generated by one object). End is right adjoint to the *free monoidal category functor* F and allegedly (at least Tony, Zbiggi and Gemini seem to agree on this), the unit of this adjuntion is an equivalence $\mathscr{D} \cong \mathsf{F}(\mathsf{End}_{\mathscr{D}})$: a monoidal category can be reconstructed from the associated (coloured) operad.

1.2.14. *Approach VI.* Finally, Florian told us how to associate to an operad O its *category of operators* $\mathsf{O}^{\otimes}$ which comes naturally with a Grothendieck opfibration whose codomain was in Florian's talk finite pointed sets. As far as I understand Shulman [9], it is in general the category of operators $(\mathsf{End}_{\mathscr{N}})^{\otimes}$ of the operad associated to $\mathscr{N}$. For $\mathscr{N} = \mathbb{N}$, this is the opposite of the simplicial category Δ, so a planar operad can be chaacterised as a Grothendieck opfibration $\mathsf{O}^{\otimes} \to \Delta^{\mathrm{op}}$ (see Lurie's collected works, e.g. [5]). The advantage of tis approach is that it is now relatively straightforward to replace categories by ∞-categories in order to define ∞-operads.

The construction of $\mathsf{O}^{\otimes}$ looks as if it could be the free monoidal category $\mathsf{F}(\mathsf{O})$, one forms forests and the fibre functor keeps track of their decompositions into trees (which in the symmetric case might be entangled and in the cartesian case even merge or diverge...), but it seems thinkgs are more subtle: one first takes the free semicartesian operd on the given one, so one upgrades $\mathcal{N}$ if necessary (not sure what happens if that was already $\mathbb{F}$), and only then takes the free semicaresian monoidal category on this semicartesian operad. So for semicartesian multicategories, $\mathsf{F}(\mathsf{O}) = \mathsf{O}^{\otimes}$ as is also stated on nLab [6].

1.3. **Operads with multiplication.**

1.3.1. *Assumption.* For the time being, $\mathcal{N} = \mathbb{N}$.

1.3.2. *Definition.* An *operad with multiplication* is an operad O together with an operad morphism $\mathsf{Ass}^{\mathbb{N}} \to \mathsf{O}$.

1.3.3. *Remark.* In $\mathscr{C} = \mathbf{Set}, \mathbf{Top}, \mathbf{Mod}_{\mathbb{K}}$, this is an operad together with an element $\mu \in \mathsf{O}_2$ such that $\mu \circ_{2,1} \mu = \mu \circ_{2,2} \mu$.

1.3.4. *Remark.* Gerstenhaber introduced this under the name *comp algebra* and called μ the *distinguished element*.

1.3.5. *Remark.* $\mathsf{Ass}^{\mathbb{N}} \to \mathsf{O}$ induces a forgetful functor from O-algebras to associative algebras. So O-algebras are associative algebras wth more structure added.

1.3.6. *Example.* By very definition, turning the endomorphism operad End_X into an operad with multiplication is the same as turning X into a monoid. This is the key example considered by Gerstenhaber in $\mathscr{C} = \mathscr{D} = \mathbf{Mod}_{\mathbb{K}}$ where X is simply a unital associative $\mathbb{K}$-algebra. However, recall the example $\mathscr{C} = \mathbf{Mod}_{\mathbb{K}}, \mathscr{D} = \mathbf{Mod}_{R^e}$ where R is a $\mathbb{K}$-algebra. Then X is a $\mathbb{K}$-algebra with a $\mathbb{K}$-algebra morphism $R \to X$ (an R-*ring*).

1.3.7. *Example.* If H is a Hopf algebra over $\mathbb{K}$ ($\mathscr{C} = \mathscr{D} = \mathbf{Mod}_{\mathbb{K}}$), define

$$\mathsf{C}(n) := \mathbf{Mod}_{\mathbb{K}}(H^{\otimes n}, \mathbb{K})$$

and for each $\varphi \in \mathsf{C}(p)$ the map

$$D_{\varphi}\colon\ H^{\otimes p} \to H, \quad h^1 \otimes \cdots \otimes h^p \mapsto \varphi(h^1_{(1)}, \ldots, h^p_{(1)}) h^1_{(2)} \cdots h^p_{(2)}.$$

Then C becomes an operad with multiplication

$$\mu = \varepsilon^H \mu^H$$

where $\mu^H\colon H \otimes H \to H$ is the multiplication in H and ε^H is its counit, and with

$$(\varphi \circ_{p,i} \psi)(h^1, \dots, h^{p+q-1})$$
$$:= \varphi(h^1, \dots, h^{i-1}, D_\psi(h^i, \dots, h^{i+q-1}), h^{i+q}, \dots, h^{p+q-1}).$$

Note that $\mu^H = D_\mu$.

1.3.8. *Remark.* C is the $\mathbb{K}$-linear dual of the (unnormalised) bar construction of H, so this carries the structure of a cosimplicial $\mathbb{K}$-module whose cohomology is $\mathrm{Ext}_H(\mathbb{K}, \mathbb{K})$ and we will link this to the opread structure in a minute. One could also work directly with the bar construction and give it a *co-operad structure* which is maybe much more pleasing and enlightening in the context of bar-cobar duality. Note also all this extends to Hopf algebroids and then includes the case of the endomorphism operad covered by Gerstenhaber [2].

2. Gerstenhaber algebras

Here I explain how (algebraic) operads become (graded) pre-Lie algebras and operads with multiplication become Gerstenhaber algebras up to homotopy.

2.1. **Pre-Lie algebras.**

2.1.1. *Assumption.* $\mathcal{N} = \mathbb{S}$, so $\mathcal{D}$ is a symmetric monoidal category and "operad" means "symmetric operad".

2.1.2. Lie → Ass$^{\mathbb{S}}$. Any associative product $\bar{\circ}$ on a $\mathbb{K}$-module X turns X into a Lie algebra with respect to the commutator $[x, y] := x \bar{\circ} y - y \bar{\circ} x$, and like all such forgetful functors between types of algebra this can be expressed in terms of a morphism of symmetric operads Lie → Ass$^{\mathbb{S}}$. We now generalise this.

2.1.3. *Definition.* A *pre-Lie* (aka *Vinberg*) *algebra* structure on $X \in \mathcal{D}$ is a binary operation $\bar{\circ}\colon X \star X \to X$ that satisfies

$$\alpha(x, y, z) = \alpha(x, z, y),$$

where $\alpha(x, y, z) := (x \bar{\circ} y) \bar{\circ} z - x \bar{\circ} (y \bar{\circ} z)$ is the *associator* of $\bar{\circ}$.

2.1.4. PreLie. Obviously, there is a symmetric operad PreLie whose algerbas are pre-Lie algebras. Just as with Lie, this only can be defined if $\mathscr{C}$ allows us to add (e.g. $\mathscr{C} = \mathbf{Mod}_{\mathbb{K}}$).

2.1.5. *Proposition.* The commutator of a pre-Lie algebra structure is a Lie algebra structure.

2.1.6. *Example.* When $\mathscr{C} = \mathbf{GrMod}_{\mathbb{K}}$, "commutator" means "graded commutator", so the true formula for the commutatur without suppressed signs is $x\bar{\circ}y - (-1)^{|x||y|}y\bar{\circ}x$. Similarly, "Lie algebra" means "graded Lie algebra", so

$$[x, y] = (-1)^{|x||y|}[y, x]$$

and there are also signs in the Jacobi identity

$$[x, [y, z]] + (-1)^{(|x|+|y|)|z|}[z, [x, y]] + (-1)^{|x|(|y|+|z|)}[y, [z, x]] = 0,$$

which can be neater written as

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||z|}[z, [x, y]] + (-1)^{|x||y|}[y, [z, x]] = 0.$$

We still suppress these signs for as long as we can but bear in mind our formulas are like graphical calculus only representing the true formulas.

2.1.7. *Remark.* The converse of 2.1.5 is not true. In particular, if α is the associator of $\bar{\circ}$, then the associator of the opposite product $x\bar{\circ}^{\mathrm{op}}y = y\bar{\circ}x$ is given by $\beta(x, y, z) = \alpha(z, y, x)$. So there is a variation of pre-Lie algebras where one demands the associator to be symmetric in the first two entries, and then the commutator is also a Lie bracket. This is related to the convention choice whether pictures are read from left to right or from right to left as mentioned in 1.3.5. As yet another example, if $\bar{\circ}$ itself is a Lie bracket, then so is the its commutator, but here Jacobi tells us that the associator is rather symmetric in x and z.

2.1.8. *Example (in honour of Julius).* Take $\mathscr{B} = \mathscr{C} = \mathscr{D} = \mathbf{Mod}_{\mathbb{K}}$. A *connection* on the vector fields on an affine scheme, that is, on the derivations $X := \mathrm{Der}_{\mathbb{K}}(A)$ of a commutative algebra $A \in \mathscr{D}$ (or more generally on a *Lie-Rinehart algebra* X over A) is a morphism

$$\nabla\colon\ X \otimes X \to X, \quad x \otimes y \mapsto \nabla_x y$$

such that $\nabla_{ax} y = a\nabla_x y$, $\nabla_x(ay) = x(a)y + a\nabla_x y$ holds for all $a \in A$ ($\nabla_x y$ is referred to as the covariant derivative of y along x with respect to the connection). The connection is *flat* if

$$\nabla_x(\nabla_y z) - \nabla_y(\nabla_x z) = \nabla_{[x,y]} z$$

and *torsionless* if

$$\nabla_x y - \nabla_y x = [x, y],$$

and one easily shows that $x\bar{\circ}y := \nabla_y x$ is a pre-Lie algebra strcture if ∇ is flat and torsion-less.

2.1.9. *Example.* This is a warm-up for our actual topic. Again, take $\mathcal{N} = \mathbb{S}, \mathscr{C} = \mathscr{D} = \mathbf{Mod}_{\mathbb{K}}$, let T be the vector space with a basis given by all rooted trees (non-planer, no ordering of vertices), and define for trees x, y

$$x\bar{\circ}y := \sum_{i \in V(x)} x \circ_i y$$

where i runs through all vertices (not just leaves!) of x and $x\circ_i y$ is obtained by attaching the root of y somewhere to the vertex i as a new branch. Then (T, $\bar{\circ}$) is the free pre-Lie algebra $\widehat{\mathsf{PreLie}}(1)$ with a single generator.

2.1.10. *Theorem* [3]. Let $H = \bigoplus_{i \geq 0} (X^{\otimes i})^{S_i}$ be cofree cocommutative conilpotent coalgebra on a vector space X. Then the pre-Lie algebra structures on X correspond bijectively to *right-sided* Hopf algebra structures on H, i.e. those for which all $\bigoplus_{i \leq n} (X^{\otimes i})^{S_i}$ are right ideals. More on this later. Maybe.

2.1.11. *Example.* The Hopf algebra arising from rooted trees has been considered by Connes–Kreimer and has been generalised to a Hopf algebra whose (co)generators are labelled by Feynman diagrams in order to formalise renormalisation in terms of Hopf algebras.

2.1.12. *The important example.* Every planar (!) operad O becomes a pre-Lie algebra with

$$\varphi\bar{\circ}\psi := \sum_{i=1}^{p} \varphi \circ_{p,i} \psi, \quad \varphi \in \mathsf{O}(p).$$

The associator $\alpha(\varphi, \psi, \chi)$ simply puts ψ and χ into all pairs of inputs of φ and sums up.

Maybe $\mathbb{N}$ is enough and we accept the Lie bracket goes down in degree?

2.1.13. *The graded viewpoint.* We have kicked this can down the road until now, but the moment has come to face this. Assume from now on $\mathcal{I} = \mathbb{Z}$, so $\mathcal{C} = \mathbf{Gr}(\mathcal{B})$, and that $\mathsf{O} \in [\mathbb{N}^{\mathrm{op}}, \mathcal{B}]$ is a nonsymmetric operad in $\mathcal{B}$. Bby very definition, this means O is an $\mathbb{N}$-graded object in $\mathcal{B}$, that is, is an object in $\mathcal{C}$ if we call the arity the degree of an operation. However, this degree is not compatible with the $\circ_{p,i}$, it is compatible with the desuspension

$$\underline{\mathsf{O}} := \mathsf{s}^{-1}\mathsf{O}$$

of O. That is, we associate to an operad in $\mathcal{B}$ an underlying object in $\mathcal{C} = \mathbf{Gr}(\mathcal{B})$ that is given by

$$\underline{\mathsf{O}}(p) := \mathsf{O}(p+1)$$

so that an n-ary operation $\varphi \in \mathsf{O}(n)$ has degree

$$|\varphi| := n - 1.$$

This works with the compositions now,

$$|\varphi \circ_{p,i} \psi| = (p+q-1) - 1 = p - 1 + q - 1 = |\varphi| + |\psi|.$$

If we have an operation $x \mapsto -x$ in $\mathcal{B}$, we may now redefine

$$\varphi \bullet_{p,i} \psi := (-1)^{|\psi|(i-1)} \varphi \circ_{p,i} \psi,$$

and I memorise this sign by keeping in mind that when evaluating $\varphi \circ_{p,i} \psi$ on $x_1 \otimes \cdots \otimes x_{p+q-1}$, the ψ has to first jump over $i-1$ tensor components. And with this sign added, the associativity rules incorporate the correct signs, which, if I do not suppress them, read

$$(\varphi \bullet_{p,i} \psi) \bullet_{p+q-1,j} \chi = \begin{cases} (-1)^{(q-1)(r-1)} (\varphi \bullet_{p,j} \chi) \bullet_{p+r-1,i+r-1} \psi & j < i, \\ \varphi \bullet_{p,i} (\psi \bullet_{q,j-i+1} \chi) & i \leq j \leq i+q+1, \\ (-1)^{(q-1)(r-1)} (\varphi \bullet_{p,j-q+1} \chi) \bullet_{p+r-1,i} \psi & i+q \leq j, \end{cases}$$

Long story short: if we start with an ungraded operad O ($\mathcal{I} = \mathbf{1}$) we may upgrade this always to a graded operad ($\mathcal{I} = \mathbb{Z}$) as long as we define the degrees as above and consider the $\bullet_{p,i}$ rather than the original $\circ_{p,i}$.

2.1.14. *Homological motivation.* If P is a projective resolution of X, then we have an exact sequence

$$0 \to P_d \to \dots \to P_0 \to X \to 0$$

where P appears to be shifted in degree by 1. This is how the same degree shift sneaks in when one takes the Yoneda appraoch to Gerstenhaber algebras ([7], [8], [1]).

2.1.15. *Theorem.* If O is an operad in $\mathscr{C}$, then $\underline{\mathrm{O}}$ is a pre-Lie algebra in $\mathbf{Gr}(\mathscr{C})$ via

$$\varphi \bar{\circ} \psi := \sum_{i=1}^{p} \varphi \bullet_{p,i} \psi .$$

2.1.16. *Definition.* We denote the indued (graded!) Lie bracket on $\underline{\mathrm{O}}$ by $\{-,-\}$ and call this the *Gerstenhaber bracket.*

2.1.17. *Example.* $\mu \bar{\circ} \mu = \mu \circ_{2,1} \mu - \mu \circ_{2,2} \mu = 0.$

2.1.18. *Remark.* In [2] we have worked with $\bar{\circ}^{op}$, hence there is a sign $(-1)^{|\varphi||\psi|}$ in front of the whole thing. This just adds signs elsewhere in the definition of a Gerstenhaber algebra to be given below. We'll see whether we can write out a consistent definition with the convention taken here.

2.1.19. *If you don't like $\underline{\mathrm{O}}$.* If one wants to avoid $\underline{\mathrm{O}}$, one can view O as a graded Lie algebra whose Lie bracket is a morphism of degree 1, meaning it maps

$$\mathrm{O}(p) \otimes \mathrm{O}(q) \to \mathrm{O}(p+q-1).$$

2.2. **The impact of μ.**

2.2.1. *Assumption.* O is an operad with multiplication μ.

2.2.2. *Definition.* The *cup product* is the binary operation

$$\mathrm{O}(p) \otimes \mathrm{O}(q) \to \mathrm{O}(p+q)$$

given by

$$\varphi \smile \psi := (\mu \circ_{p,1} \varphi) \circ_{p+1,p+1} \psi$$

2.2.3. *Definition.* The *coboundary map* on O is given by

$$d := \{\mu, -\}\colon \mathrm{O}(n) \to \mathrm{O}(n+1).$$

2.2.4. *Example.* If $\mathsf{C}(n) = \mathbf{Mod}_{\mathbb{K}}(H^{\otimes n}, \mathbb{K})$ is the operad associated to a Hopf algebra H (Example 1.3.7), then

$$(\varphi \smile \psi)(h^1, \dots, h^{p+1}) = \varphi(h^1, \dots, h^p)\psi(h^{p+1}, \dots, h^{p+q})$$

is the prouct dual to the *deconcatenation coproduct* on the bar construction. The conoundary map is the coboundary map whose cohomology is $\mathrm{Ext}_H(\mathbb{K}, \mathbb{K})$,

$$\begin{aligned}(d\varphi)(h^0, \dots, h^n) = \varepsilon(h^0)\varphi(h^1, \dots, h^n) \\ - \varphi(h^0 h^1, h^2, \dots, h^n) + \dots \\ \pm \varphi(h^0, \dots, h^{n-1})\varepsilon(h^n).\end{aligned}$$

2.2.5. *Proposition.* In general, we have $dd = 0$, since

$$\{\mu, \{\mu, \varphi\}\} + \{\varphi, \{\mu, \mu\}\} - (-1)^{|\varphi|}\{\mu, \{\varphi, \mu\}\} = 0.$$

2.2.6. *DG.* $(\mathsf{O}, \smile, d)$ is a DG algebra, $(\underline{\mathsf{O}}, \{-,-\}, d)$ is a DG Lie algebra.

2.2.7. *Eckmann-Hilton.* For $\varphi \in \mathsf{O}(p), \psi \in \mathsf{O}(q)$, we have

$$\begin{aligned}(-1)^{q-1}\varphi\bar{\circ}d\psi - (-1)^{q-1}d(\varphi\bar{\circ}\psi) + (d\varphi)\bar{\circ}\psi \\ = \varphi \smile \psi - (-1)^{pq}\psi \smile \varphi.\end{aligned}$$

One way to look at this is that if $\smile$ is graded commutative, then $\bar{\circ}$ descends to the cohomology of d. Another way to look at it is that on that cohomology, $\smile$ becomes graded commutative. Note that $\smile$ does not have to do the desuspension, it uses the arity as the degree.

2.3. **Theorem.** The cohomology V of an operad with multiplication is a *Gerstenhaber algebra*, that is, it is a graded commutative algbera and $\mathsf{s}^{-1}\mathsf{V}$ is a graded Lie algebra, and we have

$$\{\alpha \smile \beta, \gamma\} = \{\alpha, \gamma\} \smile \beta + (-1)^{|\gamma|(|\alpha|+1)}\alpha \smile \{\beta, \gamma\}.$$

2.3.1. *Braces.*

2.3.2. *Loday-Ronco again.*

2.3.3. *Kontsevich and Tamarkin.*

3. Little things

Finally, I discuss all te above in terms of the little dsks or squares operad.

3.1. **The little cubes operad.**

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